Romantic Mathematical Art: Part II

Jim Henle

The Mathematical Intelligencer

ISSN 0343-6993

Math Intelligencer
DOI 10.1007/s00283-020-09971-3

ONLINE FIRST

Springer

Romantic Mathematical Art: Part II

JIM HENLE

This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

The most beautiful thing we can experience is the mysterious. It is the source of all true art and science.
—Albert Einstein.

The mysterious is indeed beautiful. And the most beautiful mysteries are those surrounding

The Impossible.

Part I of "Romantic Mathematical Art," which appeared in the previous issue of this journal, focused on the infinite. Part II focuses on impossibility, though as noted in Part I, the two are closely related. Some of the topics covered in Part I (Penrose tiling, the hydra, hypergame, astrology, and M. C. Escher) dipped noticeably into impossibility.

The art of impossibility is singular for its history, for its philosophical interest, and for its political connections. It's all here. Well, some of it is here.

Truth

It all began, probably, with the liar paradox.

The Liar

This sentence is false.

If the sentence is true, then it's false. And if it's false, then it's true. Impossible! But it could be art.

Versions of the paradox date to the ancient Greek philosopher–poet Epimenides. Was the paradox intended to please (hence, by our criteria, art)? That's a question that can't be answered. Instead, consider the paradox of Euathlus, which came along about a century later. It seems like a lot of fun. It could be art.

The story is that Euathlus studied law under Protagoras with the understanding that payment for his education would be made as soon as he won his first case. But after his studies were completed, Euathlus didn't practice law. Protagoras grew impatient and sued his student for payment. Euathlus undertook his own defense.

You see the difficulty. If the judgment is that Euathlus must pay, it would violate the agreement, since he would have lost the case. On the other hand, if the judgment is that he doesn't have to pay, then surely he *does* have to pay, having won the case![1]

Let me say now that by "paradox" I mean a self-contradictory situation in which there appear to be good arguments on both sides. A paradox can be considered

Jim Henle, Department of Mathematics and Statistics, Burton Hall, Smith College, Northampton, MA 01063, USA. e-mail: pleasingmath@gmail.com

[1] The pleasure of this paradox continues today. A look on the web yields a variety of modern perspectives.

Published online: 26 March 2020
https://doi.org/10.1007/s00283-020-09971-3

resolved only if a flaw is found in one of the arguments or if the situation can be shown not to arise.

One answer to the liar is to argue that not every sentence has a truth value. Self-referential statements are especially prone to paradox. We could resolve these paradoxes simply by denying truth values to such statements.

Miniac

My favorite self-referential paradox is the computer Miniac, devised by the Native American mathematician Thomas Storer. It is a single coin:

Any coin will do. Miniac answers every yes-or-no question correctly, but its use needs a little explanation.

Suppose you wanted to know whether the stock market will go up tomorrow. You simply ask,

> O Miniac, will the stock market go up tomorrow?

and flip the coin. Heads means yes, tails means no. Let's say you flip it and it comes up heads. Does that mean the stock market will go up?

Not necessarily. That question was preliminary. You must ask a second question:

> O Miniac, is the truth value of your answer to this question the same as the truth value of your answer to the previous question?

and flip the coin again. Suppose now it comes up tails, meaning no. At this point, you do some logical reasoning. There are two possibilities:

Case 1: the second answer is correct, and Case 2: the second answer is incorrect.

Case 1: Since the answer (no) is correct, the truth values of the answers are not the same. And since the second answer is correct, the first answer is incorrect.

Okay, that's one case. What about the other?

Case 2: Since the answer (no) is incorrect, the truth values of the answers are the same. And since the second answer is incorrect, the first answer is also incorrect.

In both cases we find that the answer to the first question is incorrect! Thus we can say with *complete confidence* that the stock market will not go up tomorrow!

As we just saw, a tails for the second question nullifies the first answer. In a similar way, a heads for the second question verifies the first answer. Miniac, in two flips, gives you an answer that we can prove logically is correct!

Paraconsistency

There have been many philosophical attempts to deal with self-referential paradoxes. The response of mathematical logicians has been to craft axioms that don't permit self-reference. That works. I'm good with that.[2] But a group of philosophers led by Graham Priest have a different solution. They permit statements to be both true and false, that is, "paradoxical"—like the liar sentence. Logic systems permitting paradoxical statements are called "paraconsistent" logics. There are different versions, each with interesting structures and consequences. A good introduction is Priest's book *In Contradiction: A Study of the Transconsistent*.[3]

Truth in political discourse has become flexible, but it's not yet actually paradoxical.[4] If Raymond Smullyan were still with us, he might construct puzzles with knights who always tell the truth, knaves who always lie, and knerds who always speak paradoxically.[5]

Knowledge

The incompleteness theorems of Kurt Gödel show that in most mathematical systems there are statements that cannot be proved true and cannot be proved false. In other words, there are questions for which it is impossible (mathematically) to know the answer.

The Axiom of Choice

A singular question awaiting an answer is whether the axiom of choice is true. The axiom of choice was first used unconsciously in proofs about infinite sets. When it was identified, it was both shunned and celebrated. In time it produced paradoxes. The most notorious was its use by Stefan Banach and Alfred Tarski to separate a solid ball into parts that could be reassembled to form two solid balls the size of the original ball. Not long after the appearance of this paradox, Kurt Gödel showed that the axiom of choice could not be disproved. Thirty years later, Paul Cohen showed that it couldn't be proved.

Philosophers have written extensively about the problem. Is there such a thing as truth? If there is, are there ways to find it? How do we know the truth of any axiom?

The axiom of choice is grudgingly accepted today by most mathematicians, despite the appearance every few years of new and lovely paradoxes. An especially artful one is the procedure devised by Chris Hardin and Alan Taylor. They show that with the axiom of choice, it's possible to

---

[2]Wait! Every use of the pronoun "I" is self-referential, isn't it? Should I stop using "I"? We will think about this.

[3]Oxford University Press, 2nd edition, 2006.

[4]This is being written in early 2020. By the time you read this, political rhetoric may have crossed that line. Consult your local logicians.

[5]After a few weeks thinking about this, I have devised some puzzles. I'll post them on the column website, www.math.smith.edu/~jhenle/pleasingmath/.

construct a function that predicts the state of the universe given all previous states. The function is not always correct, but it is "almost always" correct (in the measure-theoretic sense).[6]

The axiom of choice also makes possible a wonderful class of puzzles, one of which appeared in an earlier column.[7] The puzzles are known as hat puzzles, because many of them can be posed as problems involving someone trying to guess the color of the hat that has been placed on their head.[8]

Here is a sample: Imagine a room with infinitely many boxes, all of them closed, one for each natural number. In each box, there is a decimal number. One by one, a hundred mathematicians enter the room. Each mathematician opens many boxes, then chooses one unopened box and predicts the number it contains. The mathematicians do not communicate at all during this exercise. Incredibly, the axiom of choice provides a strategy whereby all the predictions are correct except possibly one!

The number of mathematicians is not important, as long as it's finite. The number of boxes doesn't matter either. And the contents of the boxes could be changed to anything—sets of real numbers, sets of sets of real numbers, whatever.

Paradoxes like these are often used to argue that the axiom of choice is false. Remarkably, a graduate student at Harvard, Eliot Glazer, has recently shown that the axiom of choice isn't needed in one version of the above puzzle. If there are as many boxes, each containing a real number, as there are sets of real numbers, then a strategy exists in which, again, no more than one prediction is wrong!

The Unexpected Hanging

This popular paradox is also about knowledge. It has been written about extensively.[9] Since attitudes toward capital punishment have evolved, today it is more often framed as the paradox of the surprise exam:

A teacher tells a class that they will have a test the next week and that the day of the test will be a surprise. A group of classmates discusses the situation over lunch. One says, "It can't be Friday. If we don't get a test by Thursday, then the test will have to be on Friday, and it wouldn't be a surprise."

After a few minutes, a classmate adds, "And it can't be Thursday either. If we aren't tested by Wednesday, we'll know it's either Thursday or Friday, and since Friday has already been logically eliminated, we'll know it's Thursday, and it won't be a surprise."

Eventually, the (very bright) students conclude that logically no day is possible for the exam. It's a surprise then, when the exam comes on Monday.

The general feeling among philosophers is that the paradox has been completely explained. The only problem is, which of the many explanations is right?

First, the number of days is irrelevant. The paradox can be abbreviated to the teacher saying simply "There will be an exam tomorrow and it will be a surprise." At this point, the solution seems immediate: no one, especially the students, will believe the teacher.

Philosophers have formulated this solution using three principles of knowledge:

1. KT: If you know something, then it's true.
2. TK: We know all tautologies (logically true statements).
3. MPK: If we know A and we know that A implies B, then we know B.

The students are told E (the exam is Monday) and that they won't know E. Using the three principles, one can prove that they indeed know E, a contradiction. Hence one must either reject one of the three principles or else reject one of the premises, E, or that the students don't know E.[10] I'm happy to reject a premise.

But knowledge is tricky. Perhaps belief makes more sense here. One can do a similar deduction from E and not believing E. We don't have the first principle (some of us believe things that aren't true), so we'll need two more:

2′. TB: We believe all tautologies.
2′. MPB: If we believe A and we believe that A implies B, then we believe B.
4. CB: Our beliefs are consistent.
5. BB: If we believe A, then we believe that we believe it.

Again, this leads us to reject the premise. If BB seems a stretch, I offer DD:[11]

5′. DD: If we believe A, then we doubt that we doubt it.

Finally, the approach of philosophers Montague and Kaplan[12] reduces the paradox to the sort of self-referential statement we discussed earlier:

You do not and will not know
that this very statement is true.

Most solutions to the paradox blame the teacher.[13]

---

6 "A Peculiar connection between the axiom of choice and predicting the future," *Am. Math. Monthly* 115:2 (2008), 91–96.
7 Exactness. *Math. Intelligencer* 36:4 (2014), 98–101.
8 Wikipedia has a page on hat problems, but I recommend "An introduction to infinite hat problems," by Chris Hardin and Alan Taylor, *Math. Intelligencer* 30:4 (2008), 20–25.
9 For a start, see Martin Gardner, *The Unexpected Hanging and Other Mathematical Diversions*, University of Chicago Press, 1969.
10 This is all laid out in Tymoczko and Henle, *Sweet Reason*, 1st ed. (but not 2nd ed.), pp. 420–428, Freeman, 1995.
11 I'll put proofs of all these on the column website.
12 "A paradox regained," *Notre Dame J. Formal Logic* 1:3 (1960), 79–90.
13 As a teacher, I think this is healthy. I recommend it to my students. Like most of my recommendations, it is followed inconsistently.

Likelihood

There are numerous paradoxes of probability and statistics. We can't deal with them all. The favorite of anyone teaching Probability 101 might be the numerical fact that in a random group of 23 people, the chance that two of them have the same birthday is greater than half. That's simply the calculation

$$1-\left(\frac{364}{365}\times\frac{363}{365}\times\frac{363}{365}\cdots\frac{343}{365}\right)\approx 0.507\,.$$

The Two-Envelope Paradox

This is an elegant conundrum. Two sealed envelopes, each containing money, are prepared. One of the envelopes contains twice as much money as the other, but the amounts are unknown. The envelopes are shuffled and then distributed to two people. The recipients are then given the opportunity to trade envelopes.

One of the two might reason like this:

> Call the amount in my envelope A. If I switch, there's a 50% chance that I will gain A and a 50% chance that I will lose $A/2$. My mathematical expectation from switching is
>
> $$\frac{1}{2}(A)+\frac{1}{2}\left(-\frac{A}{2}\right)=\frac{1}{4}A.$$
>
> That's a positive expectation, so I should switch.

But the other person could reason the same way. It can't be that *both* have a positive expectation. If that were so, they might greedily spend the rest of their lives changing envelopes back and forth!

I have sent my readers to Wikipedia many times, but I think for this paradox the discussion there is excessive and confusing. Instead, consider this: First, the problem with the above reasoning is that A changes value in the computation. Suppose, for example, the sums of money are \$10 and \$20. Then the A above is sometimes \$10 and sometimes \$20. The correct reasoning is this:

> If I switch, there's a 50% chance that I have \$10 and will gain \$10 and a 50% chance that I have \$20 and will lose \$10. So my expectation is
>
> $$\frac{1}{2}(10)+\frac{1}{2}(-10)=0.$$
>
> Since the expectation for changing is 0, there is no reason to switch.

That explanation works if the probabilities are truly $\frac{1}{2}$, but that may not be the case. To have a genuine paradox, it must be the case that each person knows that no matter what their envelope contains, the expectation from switching is positive. For that we need to know the probabilities involved.

Suppose that whoever is organizing this has many possible amounts to put in the envelopes and chooses the smaller amount to be S with probability $\Pr(S)$. Then if you open your envelope and see S dollars, the smaller amount is either S or $S/2$. The probability that S is the smaller amount is

$$\frac{\Pr(S)}{\Pr(S)+\Pr(S/2)}.$$

and the probability that $S/2$ is the smaller amount is

$$\frac{\Pr(S/2)}{\Pr(S)+\Pr(S/2)}.$$

That makes the expectation

$$\frac{\Pr(S)}{\Pr(S)+\Pr(S/2)}(S)+\frac{\Pr(S/2)}{\Pr(S)+\Pr(S/2)}\left(-\frac{S}{2}\right)$$
$$=\frac{S}{\Pr(S)+\Pr(S/2)}\left(\Pr(S)-\frac{1}{2}\Pr(S/2)\right),$$

which is positive if $\Pr(S)$ is always greater than $\frac{1}{2}\Pr(S/2)$. In that case, you should always switch, and we might have a real paradox. But is it possible for $\Pr(S)$ always to be greater than $\frac{1}{2}\Pr(S/2)$?

Indeed it is! Here's a simple example. The possible amounts are $1,2,4,8,16,,$ and we set the probability that the smaller amount is 2^n to be $.3\times .7^n$. With that, $\Pr(2^n)=.3\times .7^n$ is always greater than $\frac{1}{2}\Pr(2^n/2)=\frac{1}{2}\times .3\times .7^{n-1}$, since

$$.7^n>\frac{1}{2}\times .7^{n-1}\,.$$

From this, the expectation for switching when you see 2^n is 2^{n-4}. That's always positive.

Furthermore, the probabilities are totally reasonable. They are all positive, and they add up to 1:

$$.3+.3\times .7+.3\times .7^2+\cdots=.3\times\frac{1}{1-.7}=1\,.$$

The argument that both people will improve their expectations by switching is correct!!!

Paradox?

Ah, but this, too, can be explained. That the expectations for both increase when switching is crazy—except that there is one way in which it's not crazy.

Let's look at your expectation before you're given an envelope. You find it by adding up the products of the payoffs times their probabilities:

$$E=\sum_S S\times\Pr(S)\,.$$

But we're given that $\Pr(S)$ is always greater than $\frac{1}{2}\Pr(S/2)$, so if E is finite, then

$$\sum_S\Pr(S)>\sum_S\frac{1}{2}\Pr(S/2)\,,$$

and so

$$E=\sum_S S\times\Pr(S)>\sum_S\frac{S}{2}\times\Pr(S/2).$$

But the sum over all S is the same as the sum over all $S/2$—in both cases we're just including all possible envelope amounts. So we actually have

$$E = \sum_S \Pr(S) \times S > \sum_S \frac{S}{2} \times \Pr(S/2) = E.$$

That looks impossible, but what about the example earlier. How did that work? What is the expectation when the sums are all 2^n and $\Pr(2^n) = .3 \times .7^n$? The answer is

$$\sum_{n \geq 0} \Pr(S) \times S = \sum_{n \geq 0} .3 \times .7^n \times 2^n = \sum_{n \geq 0} .3 \times (1.4)^n = \infty.$$

And that's the explanation. If your expectation is infinite and you increase it, it's still just infinite. No paradox!

And once again, the art of the impossible touches the art of infinity.

Simpson's Paradox

Simpson's paradox is more serious and more mystifying than the two-envelope paradox.[14] Something odd can happen when data are aggregated (combined). An especially clear example is that of two Yankee ballplayers, David Justice and Derek Jeter. Justice had a higher batting average than Jeter in 1995 and 1996, but when the two years are put together, Jeter had the higher average.

	1995	1996	1995 + 1996
Jeter	12/48 (.250)	183/582 (.314)	195/630 (.310)
Justice	104/411 (.253)	45/140 (.321)	149/551 (.270)

The problem infests statistical studies. Much has been written about it. The fact that it appears in the *Stanford Encyclopedia of Philosophy* tells me that interest is wide and that the phenomenon could be considered art.

The expert on Simpson's paradox is Judea Pearl, author of many papers on the subject and of the textbook *Causality*.[15] Pearl has resolved the paradox in the sense that he has algorithms for deciding, given a case, whether the aggregated data or the disaggregated data should be respected. That's an accomplishment, but it doesn't quite neuter the paradox. There is still the possibility, given almost any set of data, that a new classification, a new way to carve up the data, will arise.

For an example, take the data on the efficacy of the flu vaccine, data that appears to show that vaccination improves your chances of avoiding the flu by 40%. This is a large data set, over 300 million people. But I can divide the data into two sets in a way that might cast doubt on the conclusion. In one half, *failing* to take the vaccine improves your chances of avoiding the flu by more than 50%. In the other half, the same is true—failing to take the vaccine improves your chances of avoiding the flu by more than 50%![16]

I chose the sets explicitly to achieve this outcome. But that doesn't mean that there isn't some important factor associated with the sets. And if not, there are many other ways to divide the data to get similar results.

It is often said (and Pearl says it too) that correlation doesn't imply causation. It may be that no statistical method can prove causation.[17]

Politics

The mathematics of social choice is full of delightful bits, and many of the bits involve impossibility. The most famous of these is Arrow's impossibility theorem, about voting systems. There's also impossibility in fair division theory. But what I find most fun is the impossibility involved in apportionment. It started early and involved names you're likely to be familiar with: George Washington, Thomas Jefferson, Alexander Hamilton, John Quincy Adams, and Daniel Webster.

Given the size of the United States House of Representatives, the apportionment problem is the problem of deciding how many representatives should be given to each state. The numbers should be, as far as possible, in proportion to the populations of the states. But you can't give states fractions of representatives. A number of the founding fathers came up with apportionment methods. What I find especially amusing is that Hamilton and Jefferson, who disagreed so often on policy, even went so far as to devise different mathematical procedures for apportionment.

What can go wrong? Suppose a state ought to have, say, 7.6 representatives. Then giving the state 7 or 8 representatives seems okay. But giving it 6 or 9 seems wrong. Jefferson's method can mess up in this way. Hamilton's method can't.

Every ten years, after the census, the House gets reapportioned. What can happen with some methods is that a state with a low growth rate might gain a seat in Congress, while a state with a higher growth rate might lose one. Hamilton's method sometimes does this.

An obvious criterion for a method is that a smaller state should never get more representatives than a larger state. Neither Hamilton's nor Jefferson's method makes this mistake, but others do.

---

[14]Noticed by Karl Pearson and Udny Yule around 1900 but named for Edward H. Simpson, who rediscovered it fifty years later.

[15]Cambridge University Press, 2000, 2009.

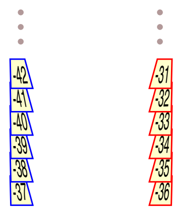
[16]I'll post the data on the website.

[17]What? You want to know whether *I* got my flu shot?
That's a personal question.
I just work here, okay?

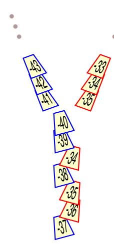
That's history. But in 1980, Michel Balinski and Peyton Young proved that every possible apportionment method is prone to at least one of these errors![18]

That Magic Trick

In my last column, I promised you an explanation of my infinite magic trick. In brief, two infinite decks comprising the positive and negative integers



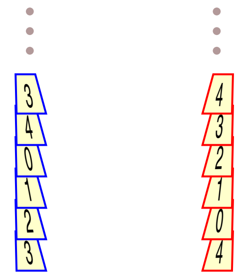
are shuffled together,



and then N cards are dealt out to each of V volunteers. The volunteers add the numbers on their cards, writing down the sum. Then the volunteers even themselves up (when one meets another and the difference between their sums is at least 2, the larger sum is reduced and the smaller sum increased by the same amount so that the difference between them is no more than 1). Surprisingly (I hope it's surprising), when all the volunteers have evened out their sums, all the sums are the same.

To show that the trick works, I have to show that the sum, S, of all the numbers on the cards dealt out is a multiple of V, so that after the exchanges, the scores will all equal S/V. That's equivalent to showing that S mod V is 0.

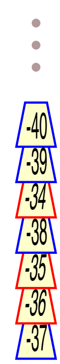
To find S mod V, we need to look at the numbers on the cards only modulo V. If we take the example in the column with, say, $V = 5$, then the decks look like this:



since $-36 \equiv 4 \bmod 5$, $(4 - (-36)) = 40$, which is divisible by 5), $-35 \equiv 0 \bmod 5$, and $-37 \equiv 3 \bmod 5$, $-35 \equiv 2 \bmod 5$.

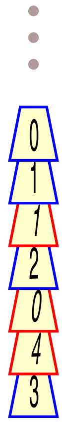
Notice that in both decks, every run of five cards has one each of every value. In the blue deck, the values are in this order: 3, 2, 1, 0, 4 (repeated indefinitely). In the red deck, the values are in the opposite order: 4 0 1 2 3 (repeated indefinitely).

The principle that Norman Gilbreath discovered is that if you shuffle together two such decks (one repeating a certain sequence of length k over and over and the other repeating the same sequence except backward), then the bottom k cards of the shuffled deck will contain the sequence (but possibly rearranged). Further, the next k cards will also contain the sequence, and the next k cards will too, and so on throughout the deck. In the example in the column,



you can see how this works.

[18]For a nice account of this, see Vicki Powers. Proportional (mis)representation: the mathematics of apportionment.'' Available at http://www.mathcs.emory.edu/~vicki/talks/Apportionment_Sept2012.pdf.



This means that in adding up the cards dealt out, we are adding up all the numbers modulo V,

$$0 + 1 + 2 + \cdots + (V - 1)\,,$$

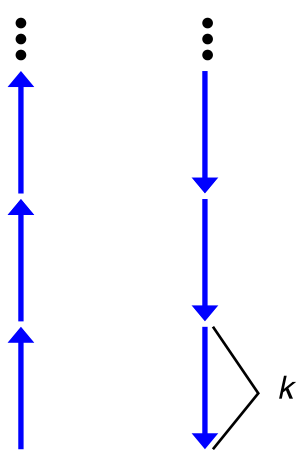
a total of N times, that is,

$$S = N(0 + 1 + 2 + \cdots + (V - 1)) = N\frac{(V-1)V}{2}\,.$$

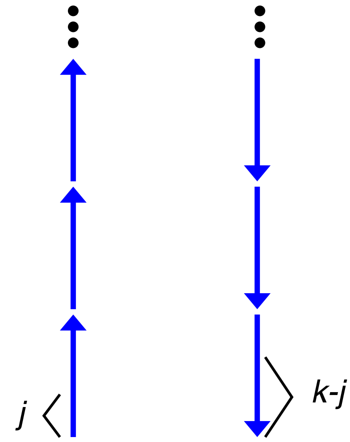
This will be a multiple of V if $V - 1$ is even or if N is even. When I perform the trick, if $V - 1$ and N are both odd, I will deal myself in, adding one to V and making $V - 1$ even. This will guarantee that S is a multiple of V, and so all volunteers end up with the same sum.

Finally, here's a proof of Gilbreath's principle in pictures:

Suppose one deck repeats a sequence of length k over and over and the other repeats the same sequence backward. I'll represent the sequence as a blue arrow. Here are the decks:

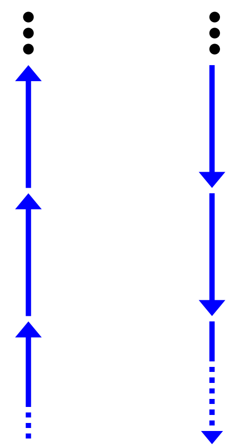


In the shuffled deck, the first k cards will contain, say, j cards from the left deck and $k - j$ cards from the the right deck:

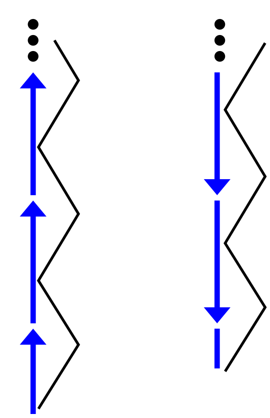


But as you can see, this means that the first k cards of the shuffled deck will have the complete sequence—though probably in a different order.

That's the first k cards of the shuffled deck. But look at what we have now:



The left deck is once again repeating a k sequence. It's a k sequence consisting of the top $k - j$ cards of the old sequence followed by the bottom j cards of the old sequence—



—and the right deck repeats the same new sequence but backward. Consequently, the reasoning that showed that the first k cards consist of all the cards in the sequence shows that the same is true for the second k cards.

And this continues throughout the deck.

Questions? Comments? I can be found at pleasingmath@gmail.com.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.