Doughnuts, Floating Ordinals, Square Brackets, and Ultraflitters

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1 Introduction

In this paper, we study partition properties of the set of real numbers. The meaning of "set of real numbers" will vary, referring at times to the collection of sequences of natural numbers, ω^ω; the collection of infinite sets of natural numbers $[\omega]^\omega$; the collection of infinite sequences of zeroes and ones, 2^ω; or $\mathcal{P}(\omega)$, the power set of ω.

The archetype for the relations is the property: "all sets of reals are Ramsey," in the notation of Erdoš and Hajnal, $\omega \to (\omega)^\omega$. This states that for every partition $F : [\omega]^\omega \to 2$, there is an infinite set $H \in [\omega]^\omega$such that F is constant on $[H]^\omega$. Like virtually all of the properties we will discuss, it contradicts the Axiom of Choice but is compatible with the principle of dependent choices (DC). DC will be used throughout the paper witout further mention.

The properties discussed in this paper will vary in two respects. Some, like $\omega \to (\omega)^\omega$, will be incompatible with the existence of an ultrafilter on ω (UF) and some will not. Some are known to be consistent relative to ZF alone, and for some, such as $\omega \to (\omega)^\omega$, the question is still open. All properties, however, are true in Solovay's model and hence are consistent relative to Con(ZF + "there exists an inaccessible cardinal") [Ma], [CS].

For details on the notation or basic properties, we refer the reader to [ER], [DPH] or [DPH1].

2 Doughnuts

Doughnut partition properties postulate the existence of homogeneous "doughnuts", sets of sets which must contain one fixed set (the hole) and be contained in another.

Definition 2.1 *If $H \subset K \in [\omega]^\omega$, $K \setminus H \in [\omega]^\omega$, then the* doughnut (H, K), *is the set $\{X : H \subseteq X \subseteq K\}$. The property, $\omega \to ((\omega))^\omega$ holds iff for all partitions $F : [\omega]^\omega \to 2$, there is a doughnut on which F is constant.*

The work of Moran and Strauss implies that $\omega \to ((\omega))^\omega$ holds for partitions into pieces which have the property of Baire [MS]. The consistency of ZF+DC+"all sets of reals have the property of Baire" was established from Con(ZF) by Shelah [Sh], the two facts together give the following.

Proposition 2.1 Con(ZF) *implies* Con(ZF + DC + $\omega \to ((\omega))^\omega$).

For completeness we prove:

Proposition 2.2 (Moran, Strauss) *For every partition of $[\omega]^\omega$ into two pieces having the property of Baire, there is a doughnut, (H, K) contained in one piece of the partition.*

Proof: Let A be a subset of $\omega^\omega$with the Baire property, and let B be $[\omega]^\omega \setminus A$. Then there are open sets W_a and W_b and meager sets M_a and M_b such that $A = W_a \Delta M_a$ and $B = W_b \Delta M_b$. Since $M_a \cup M_b$ is also meager, let $N_0, N_1, \ldots$ be a sequence of nowhere dense sets such that $M_a \cup M_b = \cup_i N_i$. We can assume that for each i, $N_i \subseteq N_{i+1}$. Let $C = \cup_{i \in \omega} \overline{N}_i$, clearly, C is meager, and so at least one of the sets $W_a \setminus C$ and $W_b \setminus C$ is non-empty and therefore residual (second category).

Identify sets in $[\omega]^\omega$with sequences of 0s and 1s in 2^ω. For any finite $s \in 2^k$, $k < \omega$, let U_s be the collection of infinite sequences with initial segment s.

Suppose $W_a \setminus C \neq \emptyset$. Let $\alpha \in W_a \setminus C$, and choose an initial segment of α, s_0, with $U_{s_0} \subseteq W_a \setminus \overline{N}_0$. Let t_0 be such that $U_{s_0 {}^\frown \{0\} {}^\frown t_0} \subseteq 2^\omega \setminus \overline{N}_1$, and let t_1 be such that $U_{s_0 {}^\frown 1 {}^\frown t_0 {}^\frown t_1} \subseteq 2^\omega \setminus \overline{N}_1$. Such t_0 and t_1 exist since $\overline{N}_1$ is closed and nowhere dense. Put $s_1 = t_0 {}^\frown t_1$. Notice that the sequences t_0, t_1 can be taken from α, (they can be the corresponding segments from α).

To simplify the writing, if $\langle s_0, \ldots, s_k \rangle$ with $k < \omega$ is a sequence of finite sequences and $r \in 2^k$, we will write $\langle s_0, \ldots, s_k \rangle * r$ to abbreviate $s_0 {}^\frown r(0) {}^\frown s_1 {}^\frown r(1) {}^\frown \ldots {}^\frown s_{k \setminus 1} {}^\frown r(k \setminus 1) {}^\frown s_k$ (we will also use the obvious generalization to infinite sequences).

Suppose we have defined s_k so that $U_{\langle s_0, \ldots, s_k \rangle * r} \subseteq 2^\omega \setminus \overline{N}_k$ hold for every $r \in 2^k$. Let $\{r_0, r_1, \ldots, r_{2^k - 1}\}$ enumerate 2^k. Define t_i for $i < 2^k$ as follows, t_0 is such that

$$U_{\langle s_0, \ldots, s_k \rangle * r_0 {}^\frown t_0} \subseteq 2^\omega \setminus \overline{N}_{k+1},$$

$$U_{\langle s_0, \ldots, s_k \rangle * r_1 {}^\frown t_0 {}^\frown t_1} \subseteq 2^\omega \setminus \overline{N}_{k+1},$$

$$\vdots$$

$$U_{\langle s_0,\ldots,s_k\rangle * r_{2^k-1}{}^\frown t_0{}^\frown t_1{}^\frown \ldots {}^\frown t_{2^k-1}} \subseteq 2^\omega \setminus \overline{N}_{k+1}.$$

And now put $s_{k+1} = t_0{}^\frown t_1{}^\frown \ldots {}^\frown t_{2^k-1}$.

In this way we obtain a sequence $s_0, s_1, \ldots$ with the property that $\langle s_0, s_1, \ldots\rangle * f \in W_a \setminus C \subseteq A$ for every $f \in 2^\omega$. The set H, then, is the set of all $n < \omega$ such that the n^{th} digit of $\langle s_0, s_1, \ldots\rangle * f$ is 1 for f the infinite sequence taking the constant value 0. The set K is the set of all $n < \omega$ such that the n^{th} digit of $\langle s_0, s_1, \ldots\rangle * f$ is 1 for f the sequence taking constant value 1. $\blacksquare_{\text{Prop. 2.2}}$

Notice that a simple modification of this proof shows that the result holds also for partitions into countably many pieces.

The property $\omega \to (\omega)^\omega$ clearly implies $\omega \to ((\omega))^\omega$. In fact, it implies a sweeping version of $\omega \to ((\omega))^\omega$.

Proposition 2.3 *$\omega \to (\omega)^\omega$ implies $\omega \to ((\omega))^\omega_{WO}$, in other words, every partition of $[\omega]^\omega$ into a well-ordered collection of classes has a homogeneous doughnut.*

Proof: Suppose $F : [\omega]^\omega \to \alpha$, for some ordinal α. Define $G : [\omega]^\omega \to 2$ by $G(p) = 0$ iff $F(p) \leq F(q)$ for all $q \in [p]^\omega$. Let $x \in [\omega]^\omega$ be homogeneous for G. The range of G must be $\{0\}$, since if $z \in [x]^\omega$ is selected so that $F(z)$ is minimal in $F''[x]^\omega$, then $G(z) = 0$.

Divide x up into the disjoint union of ω-many infinite sets $\{x_i\}_{i<\omega}$. Then since $F(\cup_{i<n} x_i) \geq F(\cup_{i<n+1} x_i)$, these must be equal for some n (or we would have an infinite descending chain). For that n, $H = \cup_{i<n} x_i$ and $K = \cup_{i<n+1} x_i$ witness doughnut homogeneity. $\blacksquare_{\text{Prop. 2.3}}$

It is not known whether $\omega \to ((\omega))^\omega$ itself implies this property, but we can prove:

Proposition 2.4 *$\omega \to ((\omega))^\omega$ implies $\omega \to ((\omega))^\omega_\omega$.*

Proof: Suppose that $F :[\omega]^\omega \to \omega$. For any set $X \subseteq \omega$ and $n < \omega$, define $F^X_n :[\omega]^\omega \to 2$ by $F^X_n(p) = 1$ iff $F(X \cup p) \geq n$. Using $\omega \to ((\omega))^\omega$, choose a doughnut (X_0, Y_0) from $[\omega]^\omega$on which $F^\emptyset_1$ is constant. If the range of F_1 is $\{0\}$, we are done. Otherwise, choose (X_1, Y_1) from $[Y_0 \setminus X_0]^\omega$ on which $F^{X_0}_2$ is constant. If the range is $\{0\}$, we are done, otherwise choose (X_2, Y_2) from $[Y_1 \setminus X_1]^\omega$ on which $F^{X_0 \cup X_1}_3$ is constant, and so on.

At some point we must have homogeneity going to $\{0\}$ since if we can continue, we will have $F(\bigcup_{i<\omega})X_i \geq n$ for all $n < \omega$. $\blacksquare_{\text{Prop. 2.4}}$

Doughnut relations are possible on other sets.

Definition 2.2 *The relation* $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ *asserts that for every* $F : \mathcal{P}(\omega) \to 2$ *there is a collection* $\{a_i\}_{i<\omega}$ *of pairwise disjoint non-empty subsets of* ω*, such that* F *is constant on all unions of these sets, i.e.,* F *is constant on* $\{\cup_{i\in b} a_i : b \subseteq \omega\}$*.*

The expression $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$ *asserts that for all* $F : P(\omega) \to 2$ *there is a collection* $\{a_i\}_{i<\omega}$ *of pairwise disjoint subsets of* ω*, such that* F *is constant on* $\{\cup_{i\in b} a_i : b \subseteq \omega, 0 \in b\}$*.*

These properties contradict the Axiom of Choice. This can be seen using the usual diagonalization argument.

Proposition 2.5 ([CS] 1.4) $ZF + AC \Rightarrow \mathcal{P}(\omega) \not\to ((\mathcal{P}(\omega)))$

Proof: Well-order the collection of all ω-collections of sets with order type $\|2^\omega\|$. Build a partition of $\mathcal{P}(\omega)$ by taking each possible homogeneous sequence in turn and defining F to be different on two unions of sets from the sequence (each containing the first set). At any stage $\alpha < \|2^\omega\|$, we have defined F on only $2 \cdot \alpha$-many sets so there are plenty of unions of the α^{th} homogeneous sequence for which F is undefined. $\blacksquare_{\text{Prop. 2.5}}$

Proposition 2.6 $\omega \to ((\omega))^\omega$ *implies* $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$

Proof: Any partition of $\mathcal{P}(\omega)$ is a partition of $[\omega]^\omega$. A homogeneous doughnut (H, K) for $[\omega]^\omega$ produces a homogeneous collection for $\mathcal{P}(\omega)$: let $a_0 = H$ and let $\{a_i\}_{0<i<\omega}$ be a partition of $K \setminus H$ into infinitely many infinite sets. $\blacksquare_{\text{Prop. 2.6}}$

While $\omega \to (\omega)^\omega$ and $\omega \to ((\omega))^\omega$ are not known to be equivalent, surprisingly, $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ and $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$ are.

Proposition 2.7 $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$ *implies* $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$

Proof: Suppose we are given $F : \mathcal{P}(\omega) \to 2$. Using $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$, obtain $\{A_n^0\}_{n<\omega}$ such that F is constant on all unions of these sets which contain A_0^0. Define F_1 on $\mathcal{P}(\omega)$ by: $F_1(p) = F\left(\bigcup_{i\in p} A_{p+1}^0\right)$. Using $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$ again, obtain $\{B_n\}_{n<\omega}$, and set for each $n < \omega$, $A_n^1 = \bigcup_{j\in B_n} A_{j+1}^0$. Now define F_2 by $F_2(p) = F\left(\bigcup_{i\in p} A_{p+1}^1\right)$ and continue in the same way.

Consider $\{A_0^k\}_{k<\omega}$. The value of F on any union X of these sets depends only on the least k such that $A_0^k \subseteq X$. This partitions ω into two pieces, $R_1 = \{k$: if k is least such that $A_0^k \subseteq X$ then $F(X) = 0\}$ and $R_2 = \{k$: if k is least such that $A_0^k \subseteq X$ then $F(X) = 1\}$. One of these is infinite, say it is R_i. Then F is constant on all unions of the sets $\{A_0^k\}_{k\in R_i}$. $\blacksquare_{\text{Prop. 2.7}}$

Proposition 2.8 *The property* $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ *implies that every subset of* ω^ω *contains or is disjoint from a perfect set.*

Proof: Given $F : \omega^\omega \to 2$, define $G : \mathcal{P}(\omega) \to 2$ as follows. If A is an infinite subset of ω, $G(A) = F(A)$ (looking at A as an increasing sequence), the definition of G on the finite subsets of ω is irrelevant. Let $A_0, A_1, \ldots$ be a sequence of pairwise disjoint infinite subsets of ω homogeneous for G, i.e. all the possible unions of elements of the sequence $A_0, A_1, \ldots$ have the same image under G. To see that this gives a perfect homogeneous set for F, just notice that the set P of all the unions of the form $A = \cup_{i\in\omega} A_{f(i)}$ with $f : \omega \to \omega$, is a set of sequences of natural numbers with no isolated points. It remains to show that it is a closed set. Let C be the complement of P. If an infinite set A of natural numbers is in C, then either there is an element $a \in A$ such that $a \notin \cup_i A_i$ or there is an i such that $A_i \cap A$ is nonempty and A_i is not contained in A. In either case, it is easy to find a neighborhood of A contained in C.

$\blacksquare_{\text{Prop. 2.8}}$

The referee has pointed out that in view of Propositions 2.3 and 2.4, the following implications follow.

1. $\omega \to ((\omega))^{\omega}{}_{\alpha}$ implies $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))_\alpha$
2. $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))_\alpha$ implies that for every $F : \mathcal{P}(\omega) \to \alpha$, there is a collection $a_{i i<\omega}$ of pairwise disjoint non-empty subsets of ω such that either $\forall b, c \in \mathcal{P} \setminus \{\emptyset\} F(\bigcup_{i\in b} a_i) = F(\bigcup_{i\in c} a_i)$, or else $\forall b, c \in \mathcal{P} \setminus \{\emptyset\} F(\bigcup_{i\in b} a_i) = F(\bigcup_{i\in c} a_i) \leftrightarrow min(b) = min(c)$
3. $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$ implies $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))_{\ \omega}$

3 Floating Ordinals

The reals ω^ω are subject to another sort of relation, infinite polarized partition relations.

Definition 3.1 $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$ *asserts that for every $F : \omega^\omega \to 2$, there is a sequence $H_0, H_1, \ldots$ of pairs of natural numbers such that F is constant on $\Pi_{i\in\omega} H_i$ (see [DPH]).*

It is not difficult to see that $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix}$ is false, that is, we cannot hope to find a homogeneous sequence of infinite sets, in fact, even $\begin{pmatrix} \omega \\ \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega \\ 2 \\ \vdots \end{pmatrix}$ fails, as the partition, $F(p) = 0$ iff $p(0) > p(1)$, witnesses. As a consequence, partition relations involving "floating omegas" were introduced.

Definition 3.2 $\begin{pmatrix}\omega\\ \omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\underbrace{\begin{matrix}\omega\\ \omega\\ \vdots\end{matrix}}\\ \\ \vdots\end{pmatrix}$ *asserts that for every $F:\omega^\omega \to 2$, there is a homogeneous sequence of non-empty sets $H_0, H_1, \ldots$ with $\{i \in \omega : |H_i| = \omega\}$ infinite.*

Since the places i in the sequence where H_i is infinite are not specified, we call these “floating omegas” (they are also placed “above the waves” in the notation).

Briefly, this relation is also false (see [DPH]), but if only a finite number of floating omegas are involved, the problem turns more interesting. The partition relation

$\begin{pmatrix}\omega\\ \omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\underbrace{\left.\begin{matrix}\omega\\ \vdots\\ \omega\end{matrix}\right\rbrace n}\\ 1\\ \vdots\end{pmatrix}$, which asserts that for every $F:\omega^\omega \to 2$, there is a homogeneous sequence of non-empty sets $H_0, H_1, \ldots$ of which at least n are infinite, is consistent, in fact, it follows from results in [MS] that it holds for Baire partitions. It is open, however, whether or not one can have several floating omegas when the rest of the sets in the homogeneous sequence are required to have at least two elements, in other words, it is not known if the partition property $\begin{pmatrix}\omega\\ \omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\underbrace{\left.\begin{matrix}\omega\\ \vdots\\ \omega\end{matrix}\right\rbrace n}\\ 2\\ \vdots\end{pmatrix}$ is consistent. The consistency of one “fixed” omega, $\begin{pmatrix}\omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\omega\\ 2\\ \vdots\end{pmatrix}$ is true [He]. Weaker properties such as $\begin{pmatrix}\omega\\ \omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\underbrace{\begin{matrix}2\\ 2\\ \vdots\end{matrix}}\\ \\ \vdots\end{pmatrix}$ were established for Baire partitions by Moran and Strauss ([MS]).

Proposition 3.1 $\begin{pmatrix}\omega\\ \omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}\underbrace{\begin{matrix}2\\ 2\\ \vdots\end{matrix}}\\ \\ \vdots\end{pmatrix}$ *implies* $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$.

Proof: In view of Proposition 2.7, we need only prove $\mathcal{P}(\omega) \to ((\mathcal{P}(\omega)))$. Let $\{X_i : i \in \omega\}$ be a collection of pairwise disjoint infinite subsets of ω. List the elements of each X_i in increasing order by $\{X_i(0), X_i(1), \dots\}$. Given $F : P(\omega) \to 2$, define $G : \omega^\omega \to 2$ by $G(\alpha) = F(\bigcup_{i\in\omega}\{X_i(k) : k \geq \alpha(i)\}_{i<\omega})$, and let $d < 2$, $H_0, H_1, \dots$ be such that $g'' \prod_{i<\omega} H_i = \{d\}$, and $M = \{i \in \omega : |H_i| = 2\}$ is infinite. We can assume that the rest of the H_i's are singletons. Let $M = \{m_0, m_1, \dots\}$.

Put $H_i = \{a_i, b_i\}$ with $a_i < b_i$ if H_i is a pair, otherwise let $H_i = \{b_i\}$.

Put $c_0 = \bigcup_{i\in\omega}\{X_i(k) : k \geq b_i\}$, and for $j > 0$, $c_j = \bigcup_{i\in\omega}\{X_{m_j}(k) : a_{m_j} \leq k < b_{m_j}\}$. Then if X is any union of the sets c_i which includes c_0, $F(X) = d$. $\blacksquare_{\text{Prop. 3.1}}$

Proposition 3.2 $\begin{pmatrix} 2 \\ 2 \\ 2 \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \smile\smile \\ \vdots \end{pmatrix}$ *is equivalent to* $\omega \to ((\omega))^\omega$.

Proof: This is easily seen by identifying the relevant spaces, 2^ω and $[\omega]^\omega$. $\blacksquare_{\text{Prop. 3.2}}$

4 Square Brackets

Most partition properties considered so far have square-bracket versions. The status of all of these is unresolved. Consider for example, $\omega \to [\omega]^\omega$.

Definition 4.1 $\omega \to [\omega]^\omega_A$ *asserts that for all partitions* $F : [\omega]^\omega \to A$, *there is an infinite* p *such that* $F''[p]^\omega \neq A$. *If* $A = \omega$, *then the subscript is omitted and we write simply* $\omega \to [\omega]^\omega$.

Proposition 4.1 (Kleinberg) $\omega \to [\omega]^\omega$ *implies* $\omega \to [\omega]^\omega_n$ *for some* $n < \omega$.

Proof: If not, then for each $n < \omega$, let F_n be a witness to the failure of $\omega \to [\omega]^\omega_n$. Then $F(p) = F_{p(0)}(p \setminus p(0))$ witnesses the failure of $\omega \to [\omega]^\omega$. $\blacksquare_{\text{Prop. 4.1}}$

Clearly, $\omega \to (\omega)^\omega$ implies $\omega \to [\omega]^\omega$. It is a long-standing open question whether $\omega \to [\omega]^\omega$ implies $\omega \to (\omega)^\omega$.

Square-bracket relations can imply round-bracket relations.

Proposition 4.2 $\omega \to [\omega]^\omega$ *implies* $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$

Proof: (We acknowledge here the help of María Carrasco). Using Kleinberg's observation let $n \in \omega$ be such that $\omega \to [\omega]^\omega_n$. Let $m \in \omega$ be such that $2^m \geq n$. We have then $\omega \to [\omega]^\omega_{2^m}$.

If $p \in [\omega]^\omega$, we denote by $(p)^m_i$ the i^{th} component of a decomposition of p into m infinite pairwise disjoint subsets obtained using a standard coding of pairs of natural numbers.

Given $F : \omega^\omega \to 2$, define $G : [\omega]^\omega \to 2^m$ by $G(p) = \langle F((p)^m_0), F((p)^m_1), \dots, F((p)^m_{m-1})\rangle$.

Let $H \in [\omega]^\omega$ be homogeneous for G (i.e., $G''[H]^\omega$ is not all of 2^m). List H in increasing order as $H = \{y_0, y_1, \dots\}$, and let J be the collection of successive pairs of elements of H, $J = \{\{y_0, y_1\}, \{y_2, y_3\}, \dots\}$. Using the same coding as above, we can decompose J into m subsequences to obtain $(J)^m_0, (J)^m_1, \dots, (J)^m_{m\smallsetminus 1}$. At least one of the $(J)^m_i$ must be a homogeneous sequence of pairs for F, this is to say, for some $i < m$, F must be constant on $Pi(J)^m_i$, the product of the pairs belonging to $(J)^m_i$. Otherwise, G takes all possible values in 2^m, because we could put together an element $p \in \Pi J$ such that the tuple $\langle F((p)^m_0), F((p)^m_1), \dots, F((p)^m_{m-1})\rangle$ is any desired sequence. $\blacksquare_{\text{Prop. 4.2}}$

5 Ultraflitters

There is a divide among partition properties between those that are consistent with UF and those that aren't. The relation, $\omega \to (\omega)^\omega$, for example, is not. The divide is useful in examining the relationship between properties.

The consistency of $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$ with UF, for example, would solve the long-open question of whether or not $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$ implies $\omega \to (\omega)^\omega$. See [LT] for results in this direction.

UF is a "choice" principle. The various proofs that certain relations are inconsistent with UF actually show inconsistency with a (possibly) weaker principle.

Definition 5.1 *A* flitter on ω *is a set* $\mathcal{F} \subseteq \mathcal{P}(\omega)$ *with the property that if* $a, b \in \mathcal{F}$, *then either* $a \cap b$ *or* $a^c \cap b^c$ *is infinite. More concisely,* $a, b \in \mathcal{F} \Rightarrow a\Delta b$ *is co-infinite.* $\mathcal{F}$ *is an* ultraflitter *if for all* $x \subseteq \omega$, *either* x *or* x^c *is in* $\mathcal{F}$.

Clearly, an ultrafilter is an ultraflitter. It can be shown that a family $\mathcal{F}$ of subsets of ω with the property $a, b \in \mathcal{F} \Rightarrow a\Delta b$ is co-infinite, is maximal if and only if it is an ultraflitter. It is well known that, viewed as a subset of 2^ω, an ultrafilter cannot be Lebesgue measurable nor can it have the property of Baire. The same holds for ultraflitters. The existence of an ultraflitter (UFL) appears to be weaker than UF, which requires at least that the intersection of members is infinite. We do not, however, have a proof of this.

[Note: Flitters are self-dual, that is, for any flitter $\mathcal{F}$, $\{x^c : x \in \mathcal{F}\}$ is also a flitter. This could be why they are called flitters.]

Like UF, UFL is a choice principle. It is equivalent to the existence of a choice function for continuum-many two-element sets. It is also equivalent to the failure of a floating object partition property.

Proposition 5.1 $\begin{pmatrix} 2 \\ 2 \\ 2 \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ \smile\!\smile \\ \vdots \end{pmatrix}$ *iff* $\neg$*UFL.*

Proof: ($\Rightarrow$) Suppose $\mathcal{F}$ is an ultraflitter. Define F on ${}^{\omega}2$ as follows. For $\alpha \in 2^{\omega}$ let $A_{\alpha} \subseteq \omega$ be such that its characteristic function is α. Given a subset $A \subseteq \omega$, let $(A)_e$ be the union of the even intervals determined by A, i.e., $(A)_e = \bigcup_{n\in\omega}[A(2n), A(2n+1))$. Put $F(\alpha) = 0$ iff $(A_{\alpha})_e \in \mathcal{F}$. If two sequences $\alpha, \alpha' \in 2^{\omega}$ differ in just one place, then the sets $(A_{\alpha})_e$ and $(A_{\alpha'})_e$ are almost complementary, and cannot be both in $\mathcal{F}$, hence $F(\alpha) \neq F(\alpha')$.

($\Leftarrow$) Suppose $F : {}^{\omega}2 \to 2$ is any partition, and suppose that no collection $\{H_i\}_{i<\omega}$ is homogeneous. For $s \subseteq \omega$, let $p_s \in {}^{\omega}2$ be defined by: $p_s(0) = 1$ iff $0 \in s$, and $p_s(i+1) = 0$ iff $\left[i \in s \Leftrightarrow i+1 \in s\right]$. For $p \in {}^{\omega}2$, let $s_p \subseteq \omega$ be defined by: $i \in s_p$ iff $\Sigma_{k=0}^{i} p(k)$ is odd. Some facts:

Define $\mathcal{F}$ by: $s \in \mathcal{F}$ iff $F(p_s) = 1$.

1. $F(p_s) \neq F(p_{s^c})$. Consider p_s and p', differing from p_s only at 0. Since no collection of homogeneous sets exists for F, $F(p_s) \neq F(p')$. But p' is actually equal to $p_{(s^c)}$

2. If s and s' are the same except that i is in s but not in s', then $F(p_s) = F(p_{s'})$. Consider p_s and $p_{s'}$. They are identical, except they differ at i and $i+1$. Form p so that it agrees with p_s everywhere except at i (and hence it agrees everywhere with $p_{s'}$ except at $i+1$). Since no collection of homogenous sets exists for F, we must have $F(p_s) \neq F(p) \neq F(p_{s'})$, and so $F(p_s) = F(p_{s'})$.

Finally, $\mathcal{F}$ must be an ultraflitter: First, if $a \subseteq \omega$, then either $a \in \mathcal{F}$ or $a^c \in \mathcal{F}$ by fact 2. Second, suppose $a, b \in \mathcal{F}$ and both $a \cap b$ and $a^c \cap b^c$ are finite. Then a and b^c differ by only a finite set. Applying fact 2 repeatedly shows that $F(p_a) = F(p_{b^c})$, so $F(p_b) = F(p_{b^c})$, contradicting fact 1. $\blacksquare_{\text{Prop. 5.1}}$

Proposition 5.2 $\begin{pmatrix} \omega \\ \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega \\ \smile\!\smile \\ \vdots \end{pmatrix}$ *implies* $\neg$*UFL.*

Proof: Any sequence $r \in {}^{\omega}\omega$ can be turned into a sequence $p_r \in {}^{\omega}2$ in the following fashion: for each n, consider $r \restriction n$. The elements in this finite sequence can be put into non-decreasing order by some permutation of n. For definiteness, consider only permutations that do not change the relative order of elements which are equal. Define $p_r(n) = 0$ iff the permutation described above is even.

Now suppose that $\mathcal{F}$ is an ultraflitter. Define $F :^{\omega} \omega \to 2$ by $F(r) = 0$ iff $\{n : p_r(n) = 0\} \in \mathcal{F}$. Suppose that $\{H_i\}_{i<\omega}$ is homogeneous for F, with H_j and H_k infinite. Take $r \in \prod_{i=0}^{\infty} H_i$. Moving $r(j)$ up or down changes the parity of some of the permutations (the values of $p_r(i)$ for $i > j$). If we change $r(j)$ to s to form r', and $r(j_1), < r(j_2) < \ldots < r(j_k)$ are the members of r between $r(j)$ and s, then for $i > j_1, j_2, \ldots, j_k$, the permutation arranging the first i elements of r in order can be amended to a permutation arranging the first i elements of r' in order by multiplying by $(r(j), r(j_1)) \cdot (r(j), r(j_2)) \cdot \ldots (r(j), r(j_k))$. Thus $p_r(i)$ changes iff k is odd.

Since H_j and H_k are infinite sets, let us suppose that $r(j) < s < t < r(k)$ and $s \in H_k$ and $t \in H_j$. Let r_1, r_2, r_3 each be the same as r with these exceptions: $r_1(j) = t$, $r_2(k) = s$, $r_3(j) = t$, $r_3(k) = s$. The difference between p_r and p_{r_1} above k is exactly the complement of the difference between p_{r_2} and p_{r_3}; in both cases we are jumping the j^{th} value, but in one case we must jump over s and the other case we don't. This leads to a contradiction. $\blacksquare_{\text{Prop. 5.2}}$

Corollary 5.1 $\begin{pmatrix} \omega \\ \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} \omega \\ \underbrace{\omega}_{} \\ \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 2 \\ 2 \\ \vdots \end{pmatrix} \to \begin{pmatrix} \underbrace{2}_{} \\ \vdots \end{pmatrix}$.

Proposition 5.3 $\omega \to [\omega]^{\omega} \Rightarrow \neg UFL$.

Proof: By Proposition 4.1, we have $\omega \to [\omega]^{\omega}_n$ for some n. We can choose n to be a prime number > 2.

For $p \in [\omega]^{\omega}$, $i < n < \omega$, let $p^m_i = \cup_{k<\omega} [p(m \cdot k + i - 1), p(m \cdot k + i)]$ (interpret $p(-1)$ as 0), where $\{p(0), p(1), \ldots\}$ are the elements of p in increasing order and we are looking at the segments determined on ω by these numbers. The segment $[p(0) \backslash p(-1)]$ is just the initial segment determined by the first element of p .

Suppose that $\mathcal{F}$ is an ultraflitter on ω, and define $F : [\omega]^{\omega} \to 2^n$ by $F(p)(i) = 1$ iff $p^n_i \in \mathcal{F}$. Let $q \in [\omega]^{\omega}$ be homogeneous for F in the sense that the range has size less than n.

Our goal is to find $q' \in [q]^{\omega}$ such that $F(q')$ is neither $\langle 0, 0, \ldots, 0\rangle$ nor $\langle 1, 1, \ldots, 1\rangle$. This will give us a contradiction, since then $F(q'), F(q' \setminus q'(0)), \ldots, F(q' \setminus q'(n-1))$ which are all rotations of $F(q')$, will be distinct (since n is prime) contradicting the homogeneity of q. To construct q', consider $\{q^{3n}_i : i < 3n\}$.

<u>Case 1</u> One of the sets $\{\bigcup_{i<n} q^{3n}_i, \bigcup_{n \le i < 2n} q^{3n}_i, \bigcup_{2n \le i < 3n} q^{3n}_i\}$ is in $\mathcal{F}$ and one isn't.

Say, for example, $\bigcup_{i<n} q^{3n}_i \in \mathcal{F}$ and $\bigcup_{i<n} q^{3n}_i \notin \mathcal{F}$. Then for any $n-2$-element subset a of $\{2n+1, 2n+2, \ldots, 3n-1\}$, $q' = \{q(3n \cdot k + 1) : i \in \{n, 2n\} \cup a, k \in \omega\}$ will serve since $F(q')$ begins $\langle 1, 0, \ldots\rangle$.

<u>Case 2</u> Either all or none of $\{\bigcup_{i<n} q^{3n}_i, \bigcup_{n \le i < 2n} q^{3n}_i, \bigcup_{2n \le i < 3n} q^{3n}_i\}$ are in $\mathcal{F}$. Suppose they are all in $\mathcal{F}$.

Case 2a One of $\{q^{3n}_{2n+2}, \dots, q^{3n}_{3n-1}\}$ is not in $\mathcal{F}$.

Then $q' = \{q(3n \cdot k + 1) : i \in \{n, 2n, 2n+2, 2n+3 \dots, 3n-1\}, k \in \omega\}$ will serve.

Case 2b Finally, if all of $\{q^{3n}_{2n+2}, \dots, q^{3n}_{3n-1}\}$ are in $\mathcal{F}$, then since $\bigcup_{i<2n} q^{3n}_i$ is not in $\mathcal{F}$ (it is the complement of $\bigcup_{2n \le i < 3n} q^{3n}_i$),

$q' = \{q(3n \cdot k + 1) : i \in \{2n, 2n+1, 2n+2, \dots, 3n-1\}, k \in \omega\}$ will serve.

$\blacksquare_{\text{Prop. 5.3}}$

We close with an example of the discriminating power of ultraflitters and ultrafilters.

Definition 5.2 *Denote by $(\omega)^\omega$ the collection of all partitions of ω into ω-many pieces. For $x \in (\omega)^\omega$, denote by $(x)^\omega$ the set of all $y \in (\omega)^\omega$ which are coarser than x (every piece of x is contained in a piece of y). The relation $\omega \leftarrow (\omega)^\omega$ asserts that for any partition F of $(\omega)^\omega$ into 2 pieces, there is an $x \in (\omega)^\omega$ such that F is constant on $(x)^\omega$.*

Carlson and Simpson show that in Solovay's model, the relation, $\omega \leftarrow (\omega)^\omega$ holds [CS].

Note that $P(\omega) \to (P(\omega))$ is equivalent, in this notation, to $\omega \leftarrow (\omega)^2$.

Proposition 5.4 Con(ZF + *"there is an inaccessible cardinal")* $\Rightarrow$ Con(ZF + $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ + *"there is an ultrafilter on ω")*.

Proof: We acknowledge here helpful remarks by Nicholas Sparks.

We work in a model of ZF+DC+$\omega \leftarrow (\omega)^\omega$ and force to add an ultrafilter. The partial ordering is the usual one, $\mathcal{P}(\omega)/\text{fin}$. We claim that $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ holds in the extension. Suppose $p \Vdash$"F is a function from $\mathcal{P}(\omega)$ to 2". For any $q \in (\omega)^\omega$, let q_n be the n^{th} piece of q, where the pieces are ordered by their least elements. Let $G : (p)^\omega \to 3$ be defined by: $G(q) = i$ iff $\{\cap q_n : n > 1\}$ forces $F(q_1) = i$. Let $r \in (p)^\omega$ be such that G is constant on $(r)^\omega$.

Notice first that the range of G on this set cannot be $\{2\}$, since there are a $j < 2$ and a set $t \subseteq \{\cap r_i : i > 1\}$ such that $t \Vdash F(r_1) = j$, and so we can form a coarsening q of r by merging all pieces r_i with $i > 1$, i not in t, into the piece r_0. Then G applied to this partition is j.

Let $\{d\}$, $d < 2$, be the range of G on $(r)^\omega$. Let $t = \{\cap r_{2i+1} : i > 1\} \subseteq p$. We claim that $t \Vdash$ "$\{r_{2i}\}_{0<i<\omega}$ is homogeneous for F." For any s, a union of these sets, we can form a coarsening q of r with s as q_1 as follows. Merge all r_{2i} which are in s. Merge all r_{2i} which are not in s with r_0, and if $2i$ is the least such that r_{2i} is in s, merge all r_{2k+1} with $2k+1 < 2i$ with r_0. We are left with a partition q where $q_1 = s$, and $\{\cap q_i : i > 1\}$ is exactly t with a finite set removed. By homogeneity, $\{\cap q_i : i > 1\}$ forces that $F(s) = d$, and the claim is proved. $\blacksquare_{\text{Prop. 5.4}}$

Corollary 5.2 Con(ZF + *there is an inaccessible cardinal*) *implies* Con(ZF + $\omega \to (\omega)^\omega$ and $\mathcal{P}(\omega) \to (\mathcal{P}(\omega))$ *are not equivalent*).

We summarize the results of this paper with a chart on the next page:

$$\omega \leftarrow (\omega)^\omega$$

$$\omega \rightarrow (\omega)^\omega$$

$$\omega \rightarrow ((\omega))^\omega_{\mathrm{WO}}$$

$$\boxed{\omega \rightarrow [\omega]^\omega \;\Leftrightarrow\; \exists n\ \omega \rightarrow [\omega]^\omega_n}$$

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} \omega \\ \omega \\ \smile\!\smile \\ \vdots \end{pmatrix}$$

$$\boxed{\omega \rightarrow ((\omega))^\omega \;\Leftrightarrow\; \omega \rightarrow ((\omega))^\omega_\omega \;\Leftrightarrow\; \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \smile\!\smile \\ \vdots \end{pmatrix}}$$

$$\boxed{\neg\mathrm{UFL} \;\Leftrightarrow\; \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ \smile\!\smile \\ \vdots \end{pmatrix}}$$

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \smile\!\smile \\ \vdots \end{pmatrix}}$$

$$\boxed{\mathcal{P}(\omega) \rightarrow (\mathcal{P}(\omega)) \;\Leftrightarrow\; \mathcal{P}(\omega) \rightarrow ((\mathcal{P}(\omega))) \Leftrightarrow\; \omega \leftarrow (\omega)^2}$$

Con(ZF+inaccessible)

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