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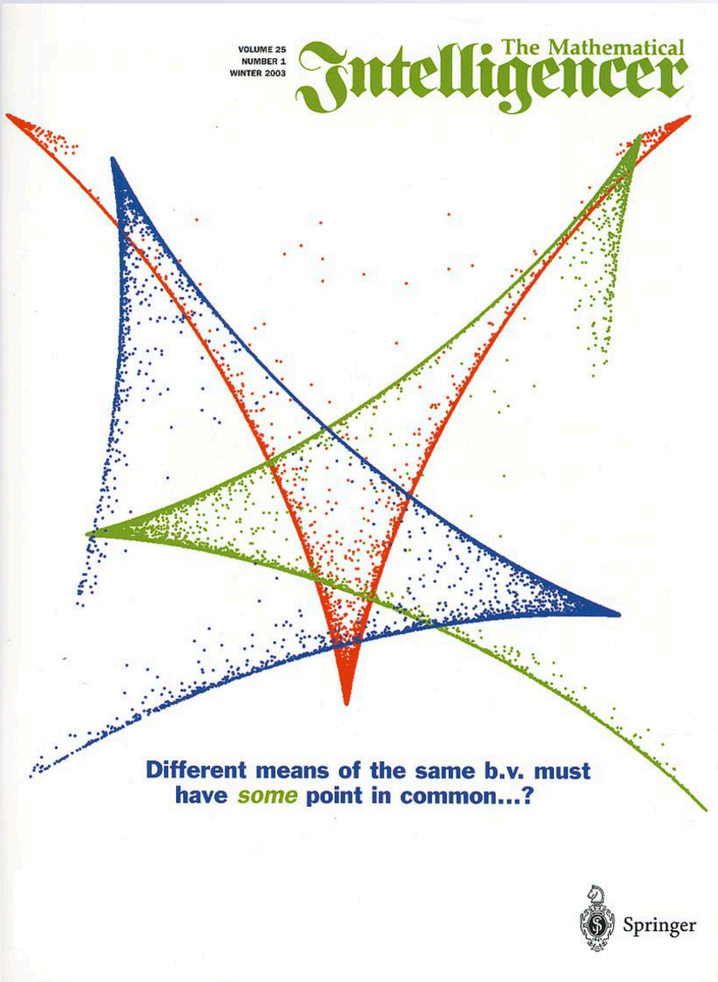

Creating Clueless Puzzles

GERARD BUTTERS, FREDERICK HENLE, JAMES HENLE, AND COLLEEN MCGAUGHEY[1]

This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

➢ Please send all submissions to the Mathematical Entertainments Editor, **Ravi Vakil,** Stanford University, Department of Mathematics, Bldg. 380, Stanford, CA 94305-2125, USA
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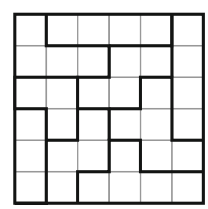
This is an invitation and a report. The report is of our efforts to create puzzles of a special kind. The invitation is to join us in that merry task.

The report will cover the history of the genre, something of the mechanics we are using, and examples of our work. As we worked (and played), the variety of clueless puzzles grew. We'll talk a little at the end about aesthetics and where the field might go.

How It Started

Of course the puzzles aren't actually clueless. The project began when one of us looked at a standard sudoku puzzle and thought there was something inelegant about the numbers. Would it be possible to create a sudoku-like puzzle in which there were no numerical clues? There would have to be something else, of course. Perhaps there could be odd-shaped regions . . .?

After some thought and fruitless attempts to create a 4×4 or a 5×5 puzzle, the first "clueless" was discovered.



The instructions are to place the digits 1, 2, 3, 4, 5, 6 in the cells to form a Latin square (no digit appearing twice in a row or column) in which the numbers in each region add to the same sum.

"Clueless" is now clear. We mean the absence of numerical clues.

The puzzle above seems impossible at first. But the sum of the numbers in any row is $1 + 2 + 3 + 4 + 5 + 6 = 21$. That means the total sum of the numbers in the entire square is $21 \times 6 = 126$. Since there are nine regions, each region must sum to $126/9 = 14$. Now look at the straight, three-cell region at the bottom left. A little thought tells you that the numbers in this region could only be 6, 5, and 3—no other triple of distinct digits sums to 14. The three-cell region just above it can't have the same three numbers. A little more thought shows you it must be 6, 4, and 4.

Thus, the numbers at the top of the left column must be 1 and 2. That tells us that the remaining numbers in the top left region must be 6 and 5.

[1]With thanks to the editor and reviewers!

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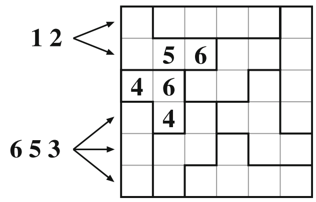


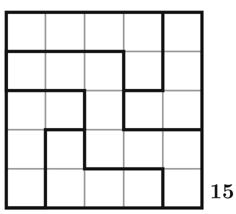
It's a pleasant task filling in the rest of the puzzle. A catalog of the four-number combinations that add to 14 is useful.[2]

All the puzzles in this paper will require taking a set of initial (non-zero) natural numbers and placing copies in a square, with all the numbers distinct in every row and column. Thus, we can actually eliminate all numbers from the description of the puzzle.

Finding Small Puzzles

We looked first for smaller puzzles of this type. Of course there is a 1×1 puzzle.

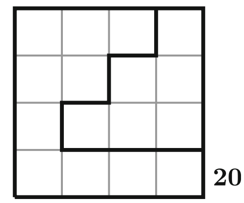
At the very least, an elegant puzzle must have a unique solution. We quickly convinced ourselves there were no 2×2 or 3×3 puzzles. We did find a 5×5 puzzle.



Now that you know the trick of finding what the sum of each region is, we write that sum (15, in this case) next to the square. But this is just because we're friends. Underneath, the puzzles are *clueless*.

We looked hard at the 4×4 square. The numbers would add to 40. In theory, if $rs = 40$ there might be a 4×4 puzzle with r regions each summing to s. The sum s must be at least 4, since 4 will be in the square. But $s = 4$ is not possible. Every region with a 3 would also have to have a 1. So any solution would have four two-cell regions with 3 and 1. Then switching the 3s and 1s would give us a second solution to the puzzle.

For a similar reason, s can't be 5. Clearly $s = 40$ is absurd. We suspected none of the others ($s = 8, \quad 10, \quad 20$) were possible (we couldn't find any). But then to our great surprise we happened upon this,



a 4×4 clueless with just two regions. Try it. There's only one solution!

We thought we found a different 6×6 clueless puzzle, one with 14 regions, each adding to 9. But then a computer program we wrote to solve puzzles found that our puzzle had more than one solution.

Larger Puzzles

The puzzles are enjoyable in themselves, but the real entertainment is finding them. They aren't easy to find. Ken Ken puzzles, which these resemble, are ubiquitous. And a grid of numbers satisfying the sudoku rules can be turned into a puzzle by adding sufficient clues. Clueless puzzles, on the other hand, are rare.

We tried writing programs to find clueless puzzles. We looked at the case of 6×6 squares with each region summing to 9. We examined all 812,851,200 Latin squares. For each square, we found all clueless partitions for which that square was a solution. We used this information to find

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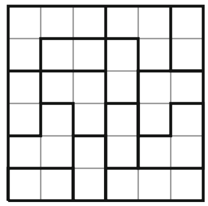
[2]The answers to all the puzzles in this paper can be found at http://www.math.smith.edu/~jhenle/cluelessanswers/

all partitions with only one solution. It took an iMac an hour to find them. There are only 640. Here's one.



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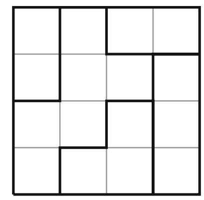
In the case of 9 regions, where the region sums are 14, it took our machine 3 days to find the 989,720 clueless boards. To resolve the 6×6 case of 7 regions adding up to 18 would take our program hundreds of years. Puzzles on the 7×7 square are even worse; there are over 61 trillion Latin squares of order 7.

Using similar methods we proved that there are no 4×4 puzzles except variations of the one we found earlier.

New Ideas

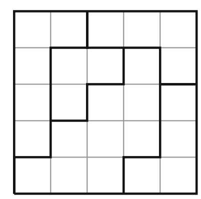
We thought of putting blanks in the square, one in each row and column. The puzzles are still clueless. In the 6×6 square, for example, each row and column would have the digits 1, 2, 3, 4, 5. We have found, so far, examples of four species of 6×6 squares:

Quite early we imagined a puzzle in which the sums of the regions, rather than being all the same, had to be all different. We haven't found a puzzle with that simple clue; getting a puzzle with a unique solution appears difficult. But with an additional condition we found something nice.



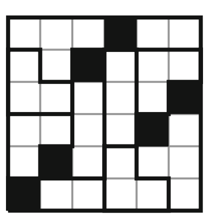
consecutive

For this puzzle, the challenge is to fill in numbers so that the region sums form a sequence of consecutive numbers. We also have a 5×5 of this sort.



consecutive

Then one of us imagined a square in which the clue was only that if one region had more cells than another, then the sum of the larger region had to be less than the sum of the smaller. We have, so far, one example.



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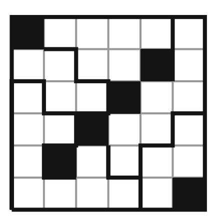

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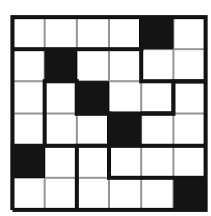

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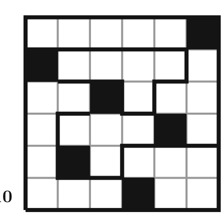

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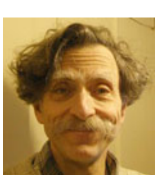

JAMES HENLE is a professor at Smith College. He has worked in set theory, geometry, nonstandard analysis, combinatorics, economics, and finite games. He is the author, with Tom Tymoczko and Jay Garfield, of *Sweet Reason: A Field Guide to Modern Logic* and, with David Cohen, *Calculus: The Language of Change.*

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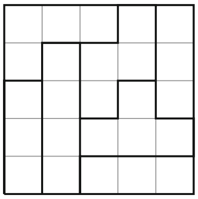
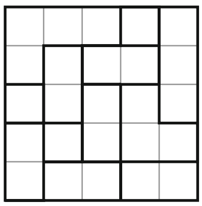


anti-monotonic

The four puzzles we showed earlier with blocks could be called "5 in 6" puzzles (the digits are 1-5 and they are set in a 6 × 6 square). That name prompted us to imagine what a "6 in 5" puzzle might be. After some fooling around, we found a 7 in 5:



7 in 5

The instructions are to place digits from 1 to 7 in the square, never using the same digit in a row or column, in such a way that the sums of the regions are all the same. There is, of course, only one solution.

Finally, there is the reverse problem. These could be called "clueful" puzzles. Here's one.

4	3	1	5	2	6
1	5	4	2	6	3
5	4	3	6	1	2
2	6	5	4	3	1
6	1	2	3	4	5
3	2	6	1	5	4

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The challenge is to divide the square into fourteen regions, each with the same sum. There is a unique answer and by happy chance it is itself a clueless puzzle.

Degrees of Elegance

The motivation for clueless puzzles was elegance. Elegance remains an issue in subtle ways.

The "anti-monotonic" puzzle was slightly inelegant in that two same-sized regions have different sums.

1	5	2	3	4
2	4	5	1	3
4	3	1	2	5
5	2	3	4	1
3	1	4	5	2

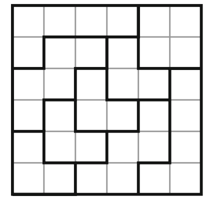
anti-monotonic

It would be nice to have a puzzle with the requirement,

$$\text{Size}(A) \leq \text{Size}(B) \Leftrightarrow \text{Sum}(A) \geq \text{Sum}(B).$$

That would guarantee that all same-sized regions have the same sum and all same-summed regions have the same size. But we haven't found a puzzle like this.

Symmetric patterns are more elegant than asymmetric patterns. There are some nice symmetric clueless puzzles.



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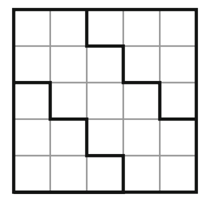
On the other hand, we found a lovely symmetric 5 × 5 which unfortunately has more than one solution. But there are only two solutions and they're symmetric.



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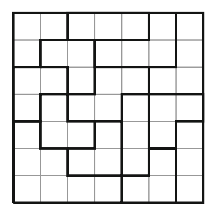
We like the puzzle, and we don't as yet know if a clueless 5 × 5 with 3 regions (and a unique answer) is possible.

Finally, one could argue that an elegant puzzle should have an elegant solution, or at least a nice, deductive path to a solution. The largest clueless puzzle we have found,



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doesn't have such a path, at least, not one we have found. The puzzle was discovered by computer. That it has a solution and a unique one was also discovered by computer.

What Next?

The reader must have questions in mind already. Of the sorts of puzzles described here, what else is possible? Are there more species of 6 × 6 puzzles, with and without blocks? Are there larger clueless puzzles?

The inventing of puzzles has just begun. What about puzzles on cylinders, Möbius strips, doughnuts, and Klein bottles? What about puzzles with more than one block per row and column?

If you have ideas, discoveries, opinions, let us know. Join us! Visit our website: http://www.math.smith.edu/~jhenle/clueless/.